# Synthesis of a wavelet transform using neural network

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# Plan of presentation

# Problem statement

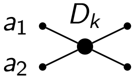
Discrete Wavelet Transform (DWT) plays an important role in signal analysis, compression and processing. Unlike other linear transforms – like DFT, DHT or DCT – DWT doesn't have one strictly defined set of basis functions. It is possible to synthesize best wavelet function suitable for particular task using adaptive methods. Especially promising are the fast multilayer linear neural networks that are able to realize wide class of linear transforms.

# Goal of research

Main goal of this research was practical verification of neural approach to wavelet synthesis, by showing that neural network is able to synthesize a new wavelet, given only transform's desired energy distribution. This led to development of a new method for unsupervised training of such multilayer network. Network with topology based on a lattice structure was used.

# Plan of presentation

# Basic element of lattice structure




$$b_1 = a_1 w_{11}^k + a_2 w_{12}^k$$
$$b_2 = a_1 w_{21}^k + a_2 w_{22}^k$$

Figure: Basic structural element of lattice structure

Operation performed by this element can be treated as a 2-by-2 matrix multiplication:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = D_k \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{, where } D_k = \begin{bmatrix} w_{11}^k & w_{12}^k \\ w_{21}^k & w_{22}^k \end{bmatrix} . \qquad (1)$$

# Lattice structure

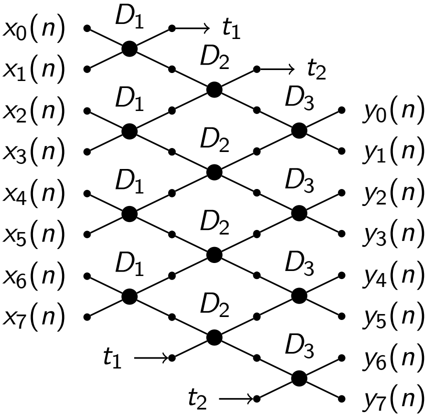



Figure: Lattice structure for realization of a wavelet transform

# Realization of inverse transform

Same structure can be used for calculating the inverse transform. To achieve this, lattice structure direction should be reversed (inputs become outputs) and transform matrices, denoted in Equation 1 as $D_k$, should be replaced with inverse matrices:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = D_k^{-1} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ , where } D_k = \begin{bmatrix} w_{11}^k & w_{12}^k \\ w_{21}^k & w_{22}^k \end{bmatrix} . \quad (2)$$

# Inverse lattice structure

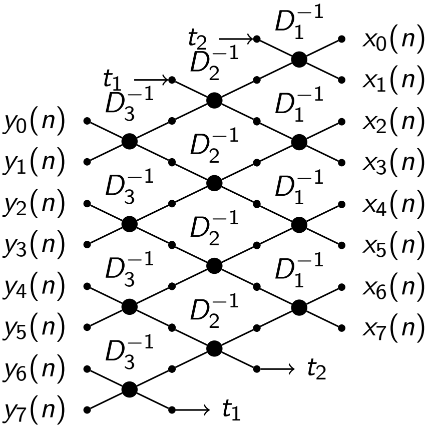



Figure: Inverse lattice structure for realization of a wavelet transform

# Orthogonal lattice structure

Let us assume that $D_k$ transform is orthogonal. By definition:

$$w_{11}^k w_{21}^k + w_{12}^k w_{22}^k = 0 \quad . \tag{3}$$

Therefore:

$$D_k \cdot D_k^T = D \quad , \tag{4}$$

where $D_k^T$ is transpose of $D_k$ matrix and $D$ is a diagonal matrix. $D_k^T$ plays a role of an inverse transform.

# Orthogonal lattice structure

Equation 3 is explicitly satisfied when:

- $w_{21} = w_{12}$ and $w_{22} = -w_{11}$. This implies that transform is symmetric:

$$S_k = S_k^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & -w_{11} \end{bmatrix} . \quad (5)$$

- $w_{21} = -w_{12}$ and $w_{22} = w_{11}$. This implies that transform is asymmetric:

$$F_k = \begin{bmatrix} w_{11} & w_{12} \\ -w_{12} & w_{11} \end{bmatrix} , \; F_k^{-1} = \begin{bmatrix} w_{11} & -w_{12} \\ w_{12} & w_{11} \end{bmatrix} . \quad (6)$$

# Neural realization of wavelet transform

Neural network was constructed with topology based on the lattice structure. Every $D_k$ base operation is replaced by a pair of linear neurons, each of them with two inputs and one output. Most important properties:

- weights of all neurons within one layer are identical,
- neurons in the network are sparsely connected.

# Training method

To synthesize a new wavelet unsupervised teaching must be used. Following criteria for teaching the neuron are proposed:

- each neuron preserves energy,
- energy ratio between the outputs of each neuron is fixed to some desired value.

# Training one layer network

Objective function for a single layer is given by formula

$$E = \sum_{j=1}^{N/2} \sum_{i=1}^{2} (d_{ji} - b_{ji}^2)^2 \quad , \tag{7}$$

where $j$ is the number of neuron in the layer, $b_{ji}^2$ is the energy of $i$–th output of a $j$–th neuron and $d_{ji}$ is the expected energy on that output. Given expected energy proportions $h$ and $g$, where $h + g = 1$, expected output values are determined: $d_{j1} = h \cdot (a_{j1}^2 + a_{j2}^2)$, $d_{j2} = g \cdot (a_{j1}^2 + a_{j2}^2)$.

# Training multilayer network

Expected energy proportion is defined only for the output layer and network is trained using backpropagation algorithm. For a straightforward determination of objective function's gradient in respect to the weights Signal Flow Graphs (SFG) are used. Due to non–standard form of objective function, adjustment of backpropagation algorithm is required. Since each output of the network is raised to the power of two before comparing it to the expected value, it is necessary to multiply error value backpropagated for each output by $-2b_{ji}$.

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# Testing method

Orthogonal neural network with topology based on orthogonal lattice structure with orthonormal symmetric base operations was used for experiments. Properties of training and testing sets:

- Signals in both sets are 16-element vectors with values taken from a row of an image (different images were used to generate each set).
- Values of vectors are normalized to range $[0, 1]$
- Training set contains 400 patterns
- Testing set contains 1000 patterns

Experiments were carried out using 4–tap, 6–tap and 8–tap transforms.

# Results

| Expected energy of low–pass outputs | Actual results | | | |
|---|---|---|---|---|
| | 4–tap transform | | 6–tap transform | |
| | training | testing | training | testing |
| 0% | 2.18% | 4.96% | 1.65% | 6.72% |
| 10% | 8.15% | 11.71% | 8.04% | 12.14% |
| 30% | 29.23% | 31.13% | 28.94% | 31.56% |
| 50% | 51.71% | 50.29% | 49.64% | 50.74% |
| 70% | 70.94% | 70.80% | 70.79% | 69.35% |
| 90% | 91.49% | 88.96% | 91.86% | 88.20% |
| 100% | 95.55% | 93.92% | 94.24% | 93.83% |
| Daubechies | 93.73% | 98.72% | 93.54% | 98.71% |

Table: Results for 4–tap and 6–tap transform

# Results

| Expected energy of low–pass outputs | Actual results 8–tap transform training | testing |
|---|---|---|
| 0% | 1.87% | 4.69% |
| 10% | 8.42% | 11.25% |
| 30% | 29.13% | 31.55% |
| 50% | 49.97% | 50.80% |
| 70% | 70.98% | 68.44% |
| 90% | 91.54% | 91.05% |
| 100% | 94.05% | 94.57% |
| Daubechies | 93.48% | 98.70% |

Table: Results for 8–tap transform

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# Conclusion

Presented neural network can be used for adaptive synthesis of a wavelet with desired energy distribution for a signal of particular class. Presented teaching method allows to effectively train multilayer network in an unsupervised learning process given only expected energy ratio between low–pass and high–pass outputs of the lattice structure. However, Daubechies wavelets still seem to offer better energy concentration than wavelets synthesized using presented method.

# Future research

Within further development of proposed orthogonal lattice structure it is necessary to determine relation between type of base operation and the class of orthogonal wavelet transforms possible to synthesize. It is also necessary to develop training methods that would allow to effectively train neural network corresponding to multilevel wavelet analysis.

# End

Questions?